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Trend to Equilibrium of Weak Solutions of the Boltzmann Equation in a Slab with Diffusive Boundary Conditions

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Recently R. Illner and the author proved that, under a physically realistic truncation on the collision kernel, the Boltzmann equation in the one-dimensional slab [0, 1] with general diffusive boundary conditions at 0 and 1 has a global weak solution in the traditional sense. Here it is proved that when the Maxwellians associated with the boundary conditions at $x=0$ and $x=1$ are the same Maxwellian M_w, then the solution is uniformly bounded and tends to M_w for $t \to \infty$.

KEY WORDS: Boltzmann equation; kinetic theory; equilibrium.

1. INTRODUCTION

In a recent paper, Cercignani and Illner[1] proved a new result on the initial-boundary value problem for the nonlinear Boltzmann equation in the interval $\Omega = [0, 1]$ in one-dimensional spatial geometry, with general diffusive boundary conditions at $x=0$ and $x=1$. The x, y, and z components of the velocity $v \in \mathfrak{R}^3$ will be denoted by ξ, η, and ζ, respectively, and the Boltzmann equation reads as follows:

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = Q(f, f) \tag{1.1}$$

with

$$Q(f, f)(x, v, t) = \iint B(n \cdot (v - v_*), |v - v_*|)(f'f'_* - ff_*) \sin\theta \, d\theta \, d\phi \, dv_* \tag{1.2}$$

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We have used the notation from refs. 1–4. For a detailed explanation of the structure of the collision term, see refs. 5–7. The angles θ and ϕ are the polar and azimuthal angles of the collision parameter $n \in S^2$ relative to a polar axis in direction $V = v - v_*$.

We assume the same truncations on the collision kernel B as in refs. 4 and 1, i.e., we suppose that there is an $\varepsilon > 0$ such that

$$B(\cdots) = 0 \qquad \text{if} \quad |v - v_*| \leqslant \varepsilon \tag{1.3a}$$

$$B \text{ is bounded} \tag{1.3b}$$

A third and less serious assumption on B is that the ratio r between

$$\int_S [n \cdot (v - v_*)]^2 \, B(n \cdot (v - v^*), |v - v_*|) \, dn$$

and

$$|v - v_*|^2 \int_S B(n \cdot (v - v^*), |v - v_*|) \, dn$$

is bounded from below.

The assumption (1.3a) can be summarized as saying that "collisions with small relative speed are disregarded" and is therefore physically more reasonable than the assumptions made in ref. 8.

For $x \in \partial\Omega$, i.e., $x \in \{0, 1\}$, and $\omega = (-1)^x$, we impose boundary conditions

$$|\xi| \, f(x, v, t) = \int_{\xi'\omega < 0} R(v' \to v; x) \, |\xi'| \, f(x, v', t) \, dv' \tag{1.4}$$

where $\xi > 0$ for $x = 0$ and $\xi < 0$ for $x = 1$. The notation is the same as in ref. 1 and consistent with that in ref. 5; in particular, ξ' denotes the x component of v'. In ref. 9, Eq. (1.4) is written more compactly as

$$f_+ = K(\cdot\,; x) \, f_- \tag{1.5}$$

where the indices $+$ and $-$ indicate that f is restricted to in- and outgoing velocities, respectively, and the kernel of the integral operator K is given by

$$K(v' \to v; x) = R(v' \to v; x) \left| \frac{\xi'}{\xi} \right| \tag{1.6}$$

(As in refs. 9 and 1, we use K to denote both the operator and its kernel.)

Finally, we have an initial value $f(x, v, 0) = f_0(x, v)$, and we shall assume that $f_0 \in L^1_+([0, 1] \times \mathfrak{R}^3)$ with the normalization

$$\iint f_0 \, dx \, dv = 1 \tag{1.7}$$

The objective of ref. 1 was to show that under reasonable assumptions on the diffuse boundary condition (1.4), and with the truncations on the collision kernel B made in (1.3), the initial-boundary value problem for the Boltzmann equation has a global weak solution in the usual sense. The main step in ref. 1 was a proof that the gain and loss terms of the collision term $Q(f, f)$, which we shall henceforth abbreviate as $G(f, f)$ and $fL(f)$, are in $L^1([0, 1] \times \mathfrak{R}^3 \times [0, T])$ for any positive time $T > 0$. Cercignani and Illner[1] also showed that the boundary conditions are satisfied as identities in the weak sense, and obtained uniform bounds on the second moment (the kinetic energy) of f.

The assumptions on the boundary kernels $R(v' \to v; x)$ here are the same as in ref. 1 and are largely identical to those made in ref. 9. Specifically, we request that

$$R \geqslant 0 \tag{1.8a}$$

$$\int_{\xi\omega > 0} R(v' \to v; x) \, dv = 1 \tag{1.8b}$$

(mass conservation)

$$\exists C_1 > 0 \qquad \text{such that} \quad \int_{\xi\omega > 0} R(v' \to v; x) \, |\xi| \, dv \geqslant C_1 \tag{1.8c}$$

("spreading condition"), and

$$\exists C_2 > 0 \qquad \text{such that} \quad \int_{\xi\omega > 0} R(v' \to v; x) \, v^2 \, dv \leqslant C_2 \tag{1.8d}$$

("energy condition").

These conditions are similar to conditions (K_0)–(K_3) in ref. 9 and exclude, as already pointed out in refs. 9 and 1, specular and reverse reflection.

Finally, in ref. 1, at variance with ref. 9, but in agreement with ref. 10 (where an extension to moving boundaries was also considered), it was required that there are two boundary Maxwellians M_0 (associated with

$x=0$) and M_1 (associated with $x=1$) which satisfy the boundary conditions at $x=0$ and $x=1$, respectively, i.e.,

$$|\xi|\, M_x(v)=\int_{\xi'\omega<0} R(v'\to v; x)\, M_x(v')\, |\xi'|\, dv' \tag{1.8e}$$

This condition is trivially satisfied for Maxwellian diffuse reflection.

The identities (1.8e) holding at $x=0$ and $x=1$ permit the use of the entropy theorem for the situation at hand.

In this paper, we shall restrict to the case when $M_0=M_1=M_w$ are the same Maxwellian $M_w(v)=\exp(-\beta\,|v|^2)$, i.e., we assume

$$|\xi|\, M_w(v)=\int_{\xi'\omega<0} R(v'\to v; x)\, M_w(v')\, |\xi'|\, dv' \tag{1.8e}$$

at both $x=0$ and $x=1$, and study the asymptotic trend of the solution for $t\to\infty$.

2. A PRIORI ESTIMATES

We now set out to prove the crucial estimates for the solution of the initial-boundary value problem and for the collision term. It is safe to assume that we deal with a sufficiently regular solution of the problem, because this can always be enforced by truncating the collision kernel and modifying the collision terms in the way described in earlier work, in particular in ref. 7. If we obtain strong enough bounds on the solutions of such truncated problems, we can then extract a subsequence converging to a renormalized solution in the sense of DiPerna and Lions; and the bounds which we do get actually guarantee that this solution is then a solution in the ordinary weak sense. Details of the limit process are given in Section 4.

In order to deal with the existence theorem in a slab at rest, with the two boundaries at the same temperature, it is convenient to remark that there is an absolute Maxwellian naturally associated with the problem, i.e., the Maxwellian M_w introduced at the end of the previous section.

A key tool is an inequality first proved by Darrozès and Guiraud[11] (see also ref. 4 or 6), which in itself is a consequence of Jensen's inequality and is based on (1.8e); it says that

$$\int \xi f \log f\, dv+\beta_w \int \xi\, |v|^2\, f\, d\xi \leqslant 0 \qquad \text{(a.e. in } t \text{ and } x=0,1) \tag{2.1}$$

Then the modified H-functional

$$H = \int f \log f \, dv \, dx + \beta \int |v|^2 f \, dv \, dx \tag{2.2}$$

will decrease in time, as a consequence of the Boltzmann equation and inequality (2.1). Thus H is bounded if bounded initially.

Let us divide the subset of $[0, 1] \times \mathfrak{R}^3$ where $f < 1$ into two subsets $\Delta^{\pm} = \{(x, v): \pm \log f < \mp \beta v^2/2)\}$. Then [since $-f \log f$ is a growing function in $(0, e^{-1})$ and less than f for $f > e^{-1}$]

$$\begin{aligned} -\int_{\Delta^+} & f \log f \, dv \, dx \\ & \leqslant \int f \, dv \, dx + \beta \int v^2 \exp(-\beta v^2/2) \, dv \, dx \leqslant C \end{aligned} \tag{2.3}$$

and in Δ^-

$$-\int_{\Delta^-} f \log f \, dv \, dx \leqslant (\beta/2) \int v^2 f \, dv \, dx \tag{2.4}$$

Then Eq. (2.2) implies that both $\int f \, |\log f| \, dv \, dx$ and $\int |v|^2 f \, dv \, dx$ are separately bounded in terms of the initial data. It is then easy to prove that the mass and entropy relations take on the following form:

$$\int f(\cdot, t) \, dv \, dx = \int f(\cdot, 0) \, dv \, dx \tag{2.5}$$

$$\begin{aligned} \int f \log f(\cdot, t) \, dv \, dx + \beta \int |v|^2 f(\cdot, t) \, dv \, dx + \int_0^t \int e(f)(\cdot, s) \, dv \, dx \, ds \\ \leqslant \int f \log f(\cdot, 0) \, dv \, dx + \beta \int |v|^2 f(\cdot, 0) \, dv \, dx \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} e(f)(x, v, t) = \tfrac{1}{4} \int_{\mathfrak{R}^3} \int_{S^+} & (f' f'_* - f f_*) \log(f' f'_* / f f_*) \\ & \times B(n \cdot (v - v_*), |v - v_*|) \, dv_* \, dn \end{aligned} \tag{2.7}$$

These estimates were first discussed by Hamdache,[12] then by Arkeryd and Cercignani[13] in the case of a general vessel (see also ref. 6).

We need some additional notation, consistent with ref. 1. For each $x \in [0, 1]$ and $t \geqslant 0$, let

$$\begin{aligned}
\rho(x, t) &= \int f(x, v, t)\, dv \\
m(t) &= \int \rho(x, t)\, dx \\
j(x, t) &= \int \xi f(x, v, t)\, dv \\
p(x, t) &= \int \xi^2 f(x, v, t)\, dv \\
q(x, t) &= \int \xi v^2 f(x, v, t)\, dv
\end{aligned} \tag{2.8}$$

We call ρ the mass density, $m(t)$ the total mass, j the mass flux (or momentum) in the x direction, p the momentum flux, and q the energy flux. At the boundaries we will need the ingoing and outgoing parts of these quantities. We use the abbreviations

$$\begin{aligned}
\rho_+ &= \int_{\xi > 0} f\, dv, \qquad & \rho_- &= \int_{\xi < 0} f\, dv \\
j_+ &= \int_{\xi > 0} \xi f\, dv, \qquad & j_- &= \int_{\xi < 0} |\xi|\, f\, dv
\end{aligned} \tag{2.9}$$

etc., such that $\rho = \rho_+ + \rho_-$, $j = j_+ - j_-$, $p = p_+ + p_-$, and $q = q_+ - q_-$.

Following the extension of the work of Bony[14] to the continuous velocity case, in refs. 1–4 the following functional was considered:

$$I[f](t) = \underbrace{\iint}_{x < y} \int_v \int_{v_*} (\xi - \xi_*)\, f(x, v, t)\, f(y, v_*, t)\, dv_*\, dv\, dx\, dy$$

where the first double integral is over the triangle $0 \leqslant x < y \leqslant 1$; they proved the relation

$$\begin{aligned}
&\int_0^T \int_0^1 \int_v \int_{v_*} (\xi - \xi_*)^2 f(x, v_*, t)\, f(x, v, t)\, dv\, dv_*\, dx\, dt \\
&\quad = I[f](0) - I[f](T) \\
&\qquad + \int_0^T \left(p(0, t) \int_0^1 \rho(x, t)\, dx + p(1, t) \int_0^1 \rho(x, t)\, dx \right) dt
\end{aligned} \tag{2.10}$$

and showed that the left-hand side of (2.10) is bounded for any finite time interval, though it may grow exponentially in time.

As remarked in ref. 1, boundedness of the left-hand side of (2.10) follows if we can obtain bounds on

$$\int_0^1 j_\pm(x, t)\,dx, \qquad \int_0^T p(0, t)\,dt, \qquad \int_0^T p(1, t)\,dt$$

Such bounds were obtained in ref. 1 by a series of estimates. Here we shall need much less; in fact we already know that the energy integral $E = \int |v|^2 f\,dv\,dx$ is bounded uniformly in time and so is the mass $[m(t) = m(0) = 1]$. Then by an elementary inequality, for every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that whenever $\zeta > 0$,

$$\xi \leqslant C(\varepsilon) + \varepsilon\xi^2 \tag{2.11}$$

Therefore, we have an estimate

$$\begin{aligned}\int_0^1 j_+(x, t)\,dx &\leqslant \int_0^1 \int_{\xi > 0} (C(\varepsilon) + \varepsilon\xi^2)\, f(x, v, t)\,dv\,dx \\ &\leqslant C(\varepsilon) + \varepsilon E\end{aligned} \tag{2.12}$$

and likewise for $\int_0^1 j_-(x, t)\,dx$.

We recall now an estimate derived in ref. 1:

$$\begin{aligned}&\int_0^t p(1, \tau)\,d\tau + \int_0^t p(0, \tau)\,d\tau \\ &\qquad\leqslant C \int_0^t E(\tau)\,d\tau + C \int_0^1 (j_+ + j_-)(x, t)\,dx \\ &\qquad\quad + C \int_0^1 (j_+ + j_-)(x, 0)\,dx\end{aligned} \tag{2.13}$$

This estimate shows that the last estimates that we needed, i.e., the uniform boundedness of $\int_0^t p(1, \tau)\,d\tau$ and $\int_0^t p(0, \tau)\,d\tau$, also follow because we have already shown that it holds for the terms in the right-hand side. We have thus proved the following result.

Lemma 2.1. If f is a sufficiently smooth solution of the initial-boundary value problem given by (1.1) and (1.4) with initial value f_0, then

$$E(t), \qquad \int_0^t (p(1, \tau) + p(0, \tau))\,d\tau, \qquad \int_0^1 (j_+ + j_-)(x, t)\,dx$$

and

$$\int_0^t \int_0^1 \int_v \int_{v_*} (\xi - \xi_*)^2 f(x, v_*, \tau) f(x, v, \tau) \, dv \, dv_* \, dx \, d\tau$$

are uniformly bounded in time in terms of the initial data.

Our objective now is to show that the collision terms themselves remain bounded. The method we employ to this end is the same as in refs. 2–4.

Following largely the notation of ref. 2, let

$$d\mu = \sin\theta \, d\theta \, d\phi \, dv_* \, dv \, dx$$

and, for $0 \leqslant \tau \leqslant T$,

$$\Delta(\tau, T) = \int_{[0,1] \times \Re^6 \times S^2 \times [\tau, T]} B(n \cdot (v - v_*), |v - v_*|) \times f(x, v, t) f(x, v_*, t) \, d\mu \, dt$$

Lemma 2.2. If the solution of the initial-boundary value problem for (1.1), (1.4) exists as a classical solution for $t \in (0, \infty)$, and if the initial value f_0 has a finite H-functional $H[f_0]$ and finite energy $E(0) = \int_0^1 \int v^2 f_0(x, v) \, dv \, dx$, then there is a constant K (depending on the initial data and ε) such that

$$\Delta(\tau, T) \leqslant K \tag{2.14}$$

The proof of Lemma 2.2 is a simple consequence of the next two lemmas.

Lemma 2.3. Let u_1 be the x component of the bulk velocity

$$u_1 = \frac{\int \xi f \, d\xi}{\int f \, d\xi} \tag{2.15}$$

Then

$$\int_{\Re^3 \times \Re^3 \times [0, T] \times \Re} (\xi - u_1)^2 f(x, v, t) f(x, v_*, t) \, dx \, dt \, dv \, dv_* < K_0 \tag{2.16}$$

where K_0 is a constant, which only depends on the initial data. In fact, the integral in (2.16) is nothing else than the integral in Lemma 2.1 (except for a factor 2) suitably rearranged. It is enough to expand the squares in both integrals and replace $\int \xi f \, d\xi$ by $u_1 \int f \, d\xi$.

We remark that the main difference with respect to ref. 1 is that we have constants in our estimates in place of functions which may grow exponentially in time. Then we can copy the proofs there, with the difference that constants will not depend on the time interval. We obtain the following result.

Lemma 2.4. Under the assumptions made in (1.3)

$$\int_{\mathfrak{R}^3 \times \mathfrak{R}^3 \times S^2 \times [0,T] \times \mathfrak{R}} |v-u|^2 f(x,v,t) f(x,v_*,t) \times B(n\cdot(v-v_*), |v-v_*|)\, dt\, dx\, dv\, dv_*\, dn < K_0 \tag{2.17}$$

where K_0 is a constant, which only depends on the initial data.

Lemma 2.5. Under the assumptions of Lemma 2.3, we have, for smooth solutions

$$\int_{\mathfrak{R}^3 \times \mathfrak{R}^3 \times S^2 \times [0,T] \times \mathfrak{R}} |v-v_*|^2 f(x,v,t) f(x,v_*,t) \times B(n\cdot(v-v_*), |v-v_*|)\, d\mu\, dt < K_0 \tag{2.18}$$

where K_0 is the same constant as in Lemma 2.4.

Lemma 2.2 now follows thanks to (2.18) and the fact that $B(\cdot,\cdot)$ is zero for $|v-v_*| \leqslant \varepsilon$. Then we have

$$\int_{\mathfrak{R}^3 \times \mathfrak{R}^3 \times S^2 \times [0,T] \times \mathfrak{R}} f(x,v,t) f(x,v_*,t) \times B(n\cdot(v-v_*), |v-v_*|)\, dt\, dx\, dv\, dv_*\, dn < K_0/\varepsilon^2 \tag{2.19}$$

3. EXISTENCE OF WEAK SOLUTIONS AND TREND TO EQUILIBRIUM

As in ref. 1, the estimates from Sections 2 imply the existence of a global weak solution for the initial-boundary value problem. This can be stated in the form of a theorem as follows.

Theorem 3.1. Let $f_0 \in L^1([0,1] \times \mathfrak{R}^3)$ be such that

$$\begin{gathered} \int f_0(\cdot)(1+|x|^2+|v|^2)\, dv\, dx < \infty \\ \int f_0\, |\ln f_0(\cdot)|\, dv\, dx < \infty \end{gathered} \tag{3.1}$$

Also assume that the collision kernel B and the boundary conditions satisfy the conditions made in Section 1. Then there is a weak solution $f(x, v, t)$ of the initial-boundary value problem (1.1), (1.4) such that $f \in C(\mathfrak{R}_+, L^1([0, 1] \times \mathfrak{R}^3))$, $f(\cdot, 0) = f_0$. This solution also satisfies the boundary conditions (1.4) a.e.

Proof. See ref. 1.

We shall now deal with the asymptotic trend for $t \to \infty$. Discussions of equilibrium states in kinetic theory are as old as the theory itself; actually these states were discussed even before the basic evolution equation of the theory, i.e., the Boltzmann equation, was formulated. The recent work on the mathematical aspects of kinetic theory has led to new results on this problem as well.

We can conjecture that the solution will tend asymptotically in time toward the nondrifting Maxwellian $M_w(v)$. A proof of this is provided by the following.

Theorem 3.2. Let f be a solution of the initial boundary value problem (1.1), (1.4). Then, when t tends to infinity, $f(\cdot, \cdot, t)$ converges strongly to the global Maxwellian $n_0 M_w$, where the constant factor n_0 is uniquely fixed by mass conservation.

Remark. The fact that the weak limit is a Maxwellian was discussed by Desvillettes[15] and Cercignani[16] (see also ref. 6), starting from a remark by DiPerna and Lions.[17] Subsequently Arkeryd[18] proved that f actually tends to a Maxwellian in a strong sense for a periodic box, but his argument works in other cases as well; his proof uses techniques of nonstandard analysis and, as such, is outside the scope of this paper. Then Lions[19] obtained the same result without resorting to nonstandard analysis. Here we shall follow the approach of ref. 19. The main differences are: (a) his assumption that $B > 0$ a.e. is not true in our case; (b) the Maxwellian will be uniquely determined.

We also point out that recently Arkeryd and Nouri[20] have sketched a proof of the fact that for boundary conditions satisfying the restriction of ref. 9 and $B > 0$, the Maxwellian is uniquely determined (for renormalized solutions). This had already been pointed out for the weak limit in ref. 16 (see also ref. 6).

Proof. It is enough to show that for every sequence t_n tending to ∞ there exists a subsequence t_{n_k} such that $f_{n_k}(x, v, t) = f(x, v, t + t_{n_k})$ converges in $L^1((0, 1) \times \mathfrak{R}^3 \times [0, T])$ to $n_0 M_w$ for any $T > 0$. The weak convergence of this sequence follows from the uniform boundedness of mass, energy, and entropy.

Thus $f_n(x, v, t) = f(x, v, t + t_n)$ is weakly compact in $L^1(\Omega \times \Re^3 \times [0, T])$ for any sequence t_n of nonnegative numbers and any $T > 0$. If $t_n \to \infty$, then there exist a subsequence t_{n_k} and a renormalized solution $M(x, v, t)$ in $L^1(\Omega \times \Re^3 \times [0, T])$ such that f_{n_k} converges weakly to $M(x, v, t)$ in $L^1(\Omega \times \Re^3 \times [0, T])$ for any $T > 0$; in addition, the gain term $Q^+(f_n, f_n)$ converges a.e. to $Q^+(M, M)$. In order to prove that M is a Maxwellian, we remark that, since the integral $\iint e(f)\, dv\, dt$ is given by (2.7) is finite, then

$$\int_{t_{n_k}}^{T + t_{n_k}} \int_{\Re^3} \int_{\Omega} \int_{S^2} \int_{\Re^3} [f(x, v', t)\, f(x, v'_*, t) - f(x, v, t)\, f(x, v_*, t)]$$

$$\times \log \frac{f(x, v', t)\, f(x, v'_*, t)}{f(x, v, t)\, f(x, v_*, t)}$$

$$\times B(n \cdot (v - v_*), |v - v_*|)\, d\mu\, dt \to 0 \qquad (k \to \infty)$$

and thus

$$\int_0^T \int_{\Re^3} \int_{\Omega} \int_{S^2} \int_{\Re^3} [f_{n_k}(x, v', t)\, f_{n_k}(x, v'_*, t)$$

$$- f_{n_k}(x, v, t)\, f_{n_k}(x, v_*, t)] \log \frac{f_{n_k}(x, v', t)\, f_{n_k}(x, v'_*, t)}{f_{n_k}(x, v, t)\, f_{n_k}(x, v_*, t)}$$

$$\times B(n \cdot (v - v_*), |v - v_*|)\, d\mu\, dt \to 0 \qquad (k \to \infty) \tag{3.1}$$

Now, according to an argument by Di Perna and Lions[17] (see also ref. 6), we can pass to the limit and obtain

$$\int_0^T \int_0^1 \int_{\Re^3} \int_{S^2} \int_{\Re^3} [M(x, v', t)\, M(x, v'_*, t) - M(x, v, t)\, M(x, v_*, t)]$$

$$\times \log \frac{M(x, v', t)\, M(x, v'_*, t)}{M(x, v, t)\, M(x, v_*, t)} B(n \cdot (v - v_*), |v - v_*|)\, d\mu\, dt = 0 \tag{3.2}$$

This implies

$$M(x, v', t)\, M(x, v'_*, t) = M(x, v, t)\, M(x, v_*, t)$$

$$(\text{a.e. in } v_*, n, x, v_*, t \text{ for } |v - v_*| \geq \varepsilon) \tag{3.3}$$

Here we have the unusual restriction on the relative speed. We use, however, the fact that one can use local arguments (in v, v_*) to deduce that $M(x, v, t)$ is a local (in x and t) Maxwellian. This was clear to

Boltzmann[21-23] for twice-differentiable functions and has been extended to the case when f is only assumed to be a distribution by Wennberg[24] Then we conclude that $M(x, v, t)$ is a local Maxwellian.

But we have for all $K > 1$

$$\begin{aligned} &|f'_{nk} f'_{nk\,*} - f_{nk}, f_{nk}| \\ &\quad \leqslant (K-1)\, f_{nk}, f_{nk} + \frac{1}{\ln K} [f_{nk}(x, v', t)\, f_{nk}(x, v'_*, t) \\ &\quad - f_{nk}(x, v, t)\, f_{nk}(x, v_*, t)] \log \frac{f_{nk}(x, v', t)\, f_{nk}(x, v'_*, t)}{f_{nk}(x, v, t)\, f_{nk}(x, v_*, t)} \end{aligned} \tag{3.4}$$

Then, since $e(f_{n_k})$ converges to 0 a.e. and $Q^+(f_{n_k}, f_{n_k})$ converges to $Q^+(M, M)$ a.e., we deduce that the loss term $Q^-(f_{n_k}, f_{n_k})$ converges a.e. to $Q^-(M, M)$. Now, the loss term is of the form $fL(f)$, where Lf is a convolution product in velocity space. Then $f_{n_k} L(f_{n_k}) \to ML(M)$ a.e.. Then either ρ_M is zero, in which case f_{n_k} converges strongly to zero (a.e. in v), or is nonzero. In the second case $L(M)$ is also nonzero and if we let $u_{n_k} = L(f_{n_k})/L(M)$, we have that $u_{n_k} \to 1$ a.e. (by the averaging lemma). Then since $f_{n_k} u_{n_k}$ tends to $M(x, v, t)$ a.e., we conclude that $f_{n_k} \to M$ a.e.

But $M(x, v, t)$ must be a (renormalized and hence weak) solution of the Boltzmann equation, or, since the collision term vanishes,

$$\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} = 0 \tag{3.5}$$

In addition M must satisfy[1] the boundary condition (1.1).

Thus the solutions of the Boltzmann equation in a slab with the boundary conditions (1.1) tend (in the case of a boundary at constant temperature) to Maxwellians satisfying the free transport equation (3.5). These Maxwellians have been well known since Boltzmann and are discussed, e.g., in Chapter III of ref. 6. Now if we specialize this general solution to the case when M depends on just the first component of r and impose the condition that $M(x, \cdot, t)$ is an L^1 function for any $t \geqslant 0$, we see that M is a Maxwellian with no drift and constant temperature; this immediately implies that M is a uniform Maxwellian, which must coincide with M_w (which is an absolute nondrifting Maxwellian) except for a factor, which is fixed by mass conservation. Thus we have proved Theorem 3.2.

5. REMARKS

The result that the solution of the Boltzmann equation converges as $t \to \infty$ strongly in $L^1[0, 1]$ to a space-homogeneous, time-independent

Maxwellian M_w whenever the latter identically satisfies the boundary conditions at each boundary point is a time-honored conjecture. Here we have proved that this is the case for the weak solutions with a truncated kernel of ref. 1.

If $M_0 \neq M_1$, it is not even clear whether a steady limit exists to which our solution can converge. The existence of steady solutions for certain boundary-value problems was proved (under more restrictive and physically unrealistic truncations on the collision kernel) in ref. 25. However, virtually nothing is known about the uniqueness and stability of these solutions.

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